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Integrable models of motion of two interacting particles in the external field

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Abstract. Classical systems with the Hamiltonian $H=\frac{1}{2}(p_1^2+p_2^2)+V_1(x_1)+V_2(x_2)+V_{12}(x_1-x_2)$ are considered, which correspond to the interaction of a system of two particles on a straight line with an external field. New sets of potentials V_1, V_2, V_{12} are found, for which these systems are integrable.

Classical integrable one-dimensional systems of interacting particles have been a subject of comprehensive study during the last ten years (a detailed review of the most important results can be found in Perelomov (1979)). There are also known integrable problems on the motion of particle systems in an external field with the Hamiltonian

$$H=\sum_{i=1}^{n}\left(\tfrac{1}{2}p_i^2+V_i(x_i)\right)+\sum_{i>j}^{n}V_{ij}(x_i-x_j). \tag{1}$$

These include the Olshanetsky and Perelomov (1976) system:

$$V_{ij}(\xi)=\alpha/\xi^2, \qquad V_i(\xi)=\beta\xi^2, \tag{2}$$

and the Adler (1977) system

$$V_{ij}(\xi)=\alpha/\sinh^2\tfrac{1}{2}\xi, \qquad V_i(\xi)=\beta\,\mathrm{e}^{\xi}. \tag{3}$$

Do other integrable systems exist with the Hamiltonian (1)? Here we present a positive answer to this question for $n=2$, when for integrability only one function K of variables p_1, p_2, x_1, x_2 is required to be found satisfying the equation

$$\{K,H\}_{\mathrm{P}}=0 \tag{4}$$

($\{\ldots\}_{\mathrm{P}}$ are the Poisson brackets).

Consider a Hamiltonian of the general form

$$H=\tfrac{1}{2}(p_1^2+p_2^2)+V(x_1,x_2)$$

and look for the function K as a polynomial in the momenta p_1, p_2:

$$K=\sum_{i=0}^{l}\sum_{m=0}^{2i}A_{im}(x_1,x_2)p_1^m p_2^{2i-m} \tag{5}$$

or

$$K=\sum_{i=0}^{l}\sum_{m=0}^{2i+1}A_{im}(x_1,x_2)p_1^m p_2^{2i-m+1}. \tag{6}$$

At $l=1$ the systems with constants of motion (5) and (6) were studied by Whittaker (1927) and Holt (1982).

Due to (4) the functions $V(x_1, x_2)$, $\{A_{im}(x_1, x_2)\}$ should obey a system of partial differential equations of first order (generally, nonlinear).

In particular, for the functions (6) this system contains l^2+3l+2 equations

$$\frac{\partial A_{l,2l}}{\partial x_1}=0, \qquad \frac{\partial A_{l,0}}{\partial x_2}=0, \qquad \frac{\partial A_{lm}}{\partial x_1}+\frac{\partial A_{l,m+1}}{\partial x_2}=0 \qquad (m=0,\dots,2l-1),$$

$$\frac{\partial A_{s,2s}}{\partial x_1}=(2s+2)A_{s+1,2s+2}\frac{\partial V}{\partial x_1}+A_{s+1,2s+1}\frac{\partial V}{\partial x_2},$$

$$\frac{\partial A_{s,0}}{\partial x_2}=A_{s+1,1}\frac{\partial V}{\partial x_1}+(2s+2)A_{s+1,0}\frac{\partial V}{\partial x_2}, \tag{7}$$

$$\frac{\partial A_{sm}}{\partial x_1}+\frac{\partial A_{s,m+1}}{\partial x_2}=(m+2)\frac{\partial V}{\partial x_1}A_{s+1,m+2}+(2s+1-m)A_{s+1,m+1}\frac{\partial V}{\partial x_2},$$

$$(s=1,\dots,l-1;\ m=0,\dots,2s-1),$$

$$\frac{\partial A_{00}}{\partial x_1}=2\frac{\partial V}{\partial x_1}A_{12}+A_{11}\frac{\partial V}{\partial x_2} \qquad ; \frac{\partial A_{00}}{\partial x_2}=A_{11}\frac{\partial V}{\partial x_1}+2A_{10}\frac{\partial V}{\partial x_2}.$$

The functions A_{lm} are polynomials in x_1 and x_2 of degree $2l$ and $(2l+1)$ for (5) and (6) respectively; the compatibility condition of equations for $A_{l-1,m}$ is a linear equation of order $2l$ or $(2l+1)$ for $V(x_1, x_2)$, coefficients of which are determined by $\{A_{lm}\}$. Upon finding its solution dependent on $2l$ (or $2l+1$) arbitrary functions of one variable, equations for the other coefficients A_{im} in (5), (6) lead either to a system of functional equations or to one functional equation for (5), $l=2$ or (6), $l=1$.

Our aim is to find integrable systems with a Hamiltonian of the type (1) at $n=2$:

$$V(x_1, x_2)=V_1(x_1)+V_2(x_2)+V_{12}(x_1-x_2). \tag{8}$$

It can be shown that for (6) at $l=1$ the above mentioned functional equation has no nontrivial solutions if we take $V(x_1, x_2)$ to be functions of the type (8). In the case of (5), $l=2$ the structure of $V(x_1, x_2)$ like (8) may appear, if we put

$$A_{20}=A_{21}=A_{23}=A_{24}=0, \qquad A_{22}=\tfrac{1}{2}.$$

Then the remaining equations (7) assume the form

$$\frac{\partial A_{12}}{\partial x_1}=\frac{\partial A_{10}}{\partial x_2}=0, \qquad \frac{\partial A_{12}}{\partial x_2}+\frac{\partial A_{11}}{\partial x_1}=\frac{\partial V}{\partial x_2}, \qquad \frac{\partial A_{10}}{\partial x_1}+\frac{\partial A_{11}}{\partial x_2}=\frac{\partial V}{\partial x_1}, \tag{9}$$

$$\frac{\partial A_{00}}{\partial x_1}=2\frac{\partial V}{\partial x_1}A_{12}+A_{11}\frac{\partial V}{\partial x_2}, \qquad \frac{\partial A_{00}}{\partial x_2}=\frac{\partial V}{\partial x_1}A_{11}+2\frac{\partial V}{\partial x_2}A_{10}. \tag{10}$$

Equations (9) can be easily solved,

$$A_{12}(x_1, x_2)=V_2(x_2), \qquad A_{10}(x_1, x_2)=V_1(x_1),$$
$$A_{11}(x_1, x_2)=\tilde{V}_{12}(x_1+x_2)-V_{12}(x_1-x_2), \tag{11}$$
$$V(x_1, x_2)=\tilde{V}_{12}(x_1+x_2)+V_{12}(x_1-x_2)+V_1(x_1)+V_2(x_2).$$

Substituting (11) into (10) we arrive at the functional equation for V_1, V_2, V_{12}, $\tilde{V}_{12}$:

$$[\tilde{V}_{12}(x_1+x_2)-V_{12}(x_1-x_2)][V_2''(x_2)-V_1''(x_1)]+2[\tilde{V}_{12}''(x_1+x_2)-V_{12}''(x_1-x_2)]$$
$$\times[V_2(x_2)-V_1(x_1)]+3\tilde{V}_{12}'(x_1+x_2)[V_2'(x_2)-V_1'(x_1)]$$
$$+3V_{12}'(x_1-x_2)[V_2'(x_2)+V_1'(x_1)]=0. \qquad (12)$$

I do not know the general solution to this equation. Some particular solutions for which all four functions V_1, V_2, V_{12}, $\tilde{V}_{12}$ are different from zero correspond to systems studied by Olshanetsky and Perelomov at $n=2$:

$$V_{12}(\xi)=\tilde{V}_{12}(\xi)=g^2P(a\xi), \qquad V_1(\xi)=V_2(\xi)=g_1^2P(a\xi)+g_2^2P(2a\xi) \qquad (13)$$

where either $g_1^2-2g^2+\sqrt{2}gg_2=0$, $g_1\neq 0$, or $g_1=0$, g, g_2 are arbitrary, and $P(a\xi)$ is the Weierstrass function.

In the case of interest for physics, $\tilde{V}_{12}=0$, all solutions can be found for a 'truncated' functional equation

$$V_{12}(x_1-x_2)[V_2''(x_2)-V_1''(x_1)]+2V_{12}''(x_1-x_2)[V_2(x_2)-V_1(x_1)]$$
$$-3V_{12}'(x_1-x_2)[V_2'(x_2)+V_1'(x_1)]=0. \qquad (14)$$

Indeed, one may introduce new variables into (12),

$$\tau=\tfrac{1}{2}(x_1-x_2), \qquad \rho=-\tfrac{1}{2}(x_1+x_2),$$

and new unknown functions connected with V_1, V_2, V_{12} by

$$V_1(x_1)=\frac{\mathrm{d}}{\mathrm{d}\tau}N(\tau-\rho), \qquad V_2(x_2)=\frac{\mathrm{d}}{\mathrm{d}\tau}L(-\tau-\rho),$$
$$V_{12}(x_1-x_2)=\left[\frac{\mathrm{d}\eta(\tau)}{\mathrm{d}\tau}\right]^{-2}. \qquad (15)$$

The functional equation (14) can be triply integrated, which reduces it to the functional equation without derivatives:

$$L(\tau+\rho)-N(\tau-\rho)=c_1(\rho)\eta^2(\tau)+c_2(\rho)\eta(\tau)+c_3(\rho) \qquad (16)$$

with c_1, c_2, c_3 arbitrary functions of ρ. Expanding both sides of (16) in a power series in ρ we arrive at an infinite system of ordinary differential equations

$$L(\tau)-N(\tau)=c_1(0)\eta^2(\tau)+c_2(0)\eta(\tau)+c_3(0),$$
$$(\mathrm{d}/\mathrm{d}\tau)^k[L(\tau)-(-1)^kN(\tau)]=c_1^{(k)}(0)\eta^2(\tau)+c_2^{(k)}(0)\eta(\tau)+c_3^{(k)}(0), \qquad k=1,2,\ldots.$$

For $L(\tau)\neq N(\tau)$ one can obtain an equation for the function $\eta(\tau)$ by eliminating $L(\tau)$ and $N(\tau)$ from the first, third and fifth equations of the system; for $L(\tau)=N(\tau)$ one uses the second, fourth and sixth equations. Upon these simple calculations we find that solutions to (16) do exist provided that the function $\eta(\tau)$ satisfies one of the two equations

$$\eta'^2=d_1(\eta-\eta_0)^2+d_2(\eta-\eta_0)+d_3$$

or

$$\eta'^2=d_1(\eta-\eta_0)^2+d_2+d_3(\eta-\eta_0)^{-2}.$$

Thus, there exist two sets of solutions of (16) for which $\eta(\tau)=a\cosh(\beta\tau+\gamma)+b$ or $\eta(\tau)=[a\cosh(\beta\tau+\gamma)+b]^{1/2}+\eta_0$, where a, b, β, γ, η_0 are arbitrary constants.

Equation (14) also has only two sets of solutions:

$$V_1(x_1) = \lambda_1 \cosh(2\beta x_1 + \gamma_1 + 2\Delta) + \lambda_2 \cosh(\beta x_1 + \gamma_2 + \Delta),$$
$$V_2(x_2) = \lambda_1 \cosh(2\beta x_2 + \gamma_1 - 2\Delta) + \lambda_2 \cosh(\beta x_2 + \gamma_2 - \Delta), \quad (17a)$$
$$V_{12}(x_1 - x_2) = \lambda_3[\sinh(\tfrac{1}{2}\beta(x_1 - x_2) + \Delta)]^{-2},$$

and

$$V_1(x_1) = \lambda_1 \cosh(\beta x_1 + \gamma_1 + \Delta), \qquad V_2(x_2) = \lambda_1 \cosh(\beta x_2 + \gamma_1 - \Delta),$$
$$V_{12}(x_1 - x_2) = \lambda_2[\sinh(\tfrac{1}{2}\beta(x_1 - x_2) + \Delta)]^{-2} + \lambda_3[\sinh(\tfrac{1}{4}\beta(x_1 - x_2) + \tfrac{1}{2}\Delta)]^{-2}, \quad (17b)$$

where β, λ_1, λ_2, λ_3, γ_1, γ_2, Δ are arbitrary constants. By a limiting procedure, one may obtain from the solutions (17*a*, *b*) both the Olshanetsky–Perelomov system (2) and the Adler system (3) at $n = 2$.

Particular solutions to the initial functional equation (12) may be pointed out, which have a structure similar to (17*a*, *b*):

$$V_1(\xi) = V_2(\xi) = \lambda_1 \cosh(\beta\xi), \qquad V_{12}(\xi) = \frac{\lambda_2}{\sinh^2(\beta\xi/2)} + \frac{\lambda_3}{\sinh^2(\beta\xi/4)},$$
$$\tilde{V}_{12}(\xi) = \lambda_4/\sinh^2(\beta\xi/2) + \lambda_5/\sinh^2(\beta\xi/4), \quad (18a)$$

$$V_1(\xi) = V_2(\xi) = \lambda_1 \cosh(\beta\xi) + \lambda_2 \cosh(2\beta\xi),$$
$$V_{12}(\xi) = \lambda_3/\sinh^2(\beta\xi/2), \qquad \tilde{V}_{12}(\xi) = \lambda_4/\sinh^2(\beta\xi/2). \quad (18b)$$

The existence of solutions to (12) different from (13), (17)–(18) remains still an open problem.

In my opinion, a point of interest for physics is the following problems, the analysis of which is beyond the scope of this note:

(i) determination of the motion of particles in systems of the type (17*a*, *b*);

(ii) construction of the Lax pairs similar to the known ones in the limiting cases (2), (3) at $n = 2$.

Systems defined by the potentials (17*a*) admit an extension to systems of n interacting particles in the external field. The Lax pairs for these systems at $n = 2$ consist of (4×4) matrices for $\gamma_1 = \gamma_2$ or of (8×8) matrices for $\gamma_1 \neq \gamma_2$. These results will be published elsewhere.

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